ISRAEL JOURNAL OF MATHEMATICS, Vol. 42, No. 4, 1982

A SPECIAL FAMILY OF ERGODIC FLOWS AND THEIR $\bar{d}$-LIMITS

BY
KYEWON KOH PARK

ABSTRACT
A flow built under a step function with a multi-step Markov partition on the base is a direct product of a Bernoulli flow with a finite rotation. A $\bar{d}$-limit of the flows in this family cannot have two irrationally related rotation factors. $\bar{d}$-closure of this family is shown to consist of all direct products of Bernoulli flows and flows of rational pure point spectrum with respect to some number.

0. Introduction

A multi-step Markov process is known to be a direct product of a Bernoulli process with a finite rotation [1]. And $\bar{d}$-limits of these processes consist of all direct products of Bernoulli processes with zero entropy processes of rational pure point spectrum [8].

We proved that flows, each of which is built under a step function with a 'straight' partition, form a dense subset of all ergodic flows in $\bar{d}$-topology [7]. Given a flow built under a step function, we approximate the base process by multi-step Markov processes in distribution and entropy. The flow built under the same step function with the n-step Markov approximation on the base is called the canonical n-step Markov approximation to the given flow.

In the first section, we thoroughly investigate the flows built under step functions with multi-step Markov partitions on the bases. We prove that every flow in this family is a direct product of a Bernoulli flow with a finite rotation flow. By putting some conditions on values of functions and/or partitions on bases, we have flows without rotation factors, i.e. Bernoulli flows. The finite rotation factor is always spanned by finite continuous names. We call the period of a rotation factor a rotation number. This number determines the corresponding zero entropy factor uniquely. Hence every flow in this family is uniquely

Received March 15, 1981 and in revised form December 10, 1981

determined by its entropy and its rotation number. Since we are interested in the flows built under step functions with multi-step Markov partitions on bases, the only flows relevant to our topic are those direct products whose zero entropy factors are rotations; i.e. its zero entropy factor has rational pure point spectrum with respect to its rotation number.

In the second section, we classify the flows which are $\bar{d}$-limits of the flows in this family. A. Fieldsteel [2] developed the relative isomorphism theory for ergodic flows adapting Thouvenot's ideas for processes [10]. One of his results is that the class of direct products of Bernoulli flows and flows of zero entropy is $\bar{d}$-closed. He also showed that this class is closed under taking factors. We show that a $\bar{d}$-limit of the flows in this special family has the zero entropy factor of rational pure point spectrum w.r.t. some number. Also the limit cannot have two rotation factors whose rotation numbers are irrationally related.

We will assume that every flow $(S_t, \bar{P}, \Omega)$ is ergodic and it is built under a step function f. Let $\bar{P} = \{\bar{P}_1, \cdots, \bar{P}_n\}$ be a partition of a flow. A partition $\bar{P}$ is called *straight*, if for every point x in the base, $(x, t_0) \in \bar{P}_i$ for some $t_0 < f(x)$ implies $(x, t) \in \bar{P}_i$ for all $0 \leqq t < f(x)$. If a partition $\bar{P}$ of a flow is straight, we denote the base with the corresponding partition by (T, P, X). We always deal with a flow with a straight partition. If the corresponding base process (T, P, X) is multi-step Markov, then we say that the flow $(S_t, \bar{P}, \Omega)$ is built under a step function with a multi-step Markov partition on the base. By $(a, b) = 1$, we mean two integers, a and b, are relatively prime. Also the g.c.d. is used for the greatest common divisor.

I am very thankful to Professor D. S. Ornstein and D. Rudolph under whose guidance this work was done.

1. In this section we consider flows, each built under a step function with a multi-step Markov partition on the base.

Let $(S_t, \bar{P}, \Omega)$ be a flow in this family with positive entropy. Since any multi-step Markov partition can be refined to a one-step Markov partition in a very natural way, it is enough to consider a flow with a one-step Markov partition on a base. Let $\{\alpha_1, \cdots, \alpha_n\}$ be the set of values of f with corresponding partition $\{P_1, \cdots, P_n\}$ on the base. Divide P_1 into $\{P_{1,(w_1,\cdots,w_k)}\}$ where

$$P_{1,(w_1,\cdots,w_k)} = \{x \in P_1 \mid T^1(x) \in P_{w_1}, T^2(x) \in P_{w_2}, \cdots, T^k(x) \in P_{w_k}, T^{k+1}(x) \in P_1\}$$

where $w_i \neq 1$ for all $i = 1, \cdots, k$.

Build the flow over P_1 as a base under a function f' where

$$f'(x) = \sum_{i=0}^{k} f(T^i(x)) \qquad \text{if } x \in P_{1,(w_1,\cdots,w_k)}.$$

Let T_r denote the induced transformation of T on P_1. We call P_1 a *base set* and $(w_1, \cdots, w_k)$ a *path name* of $x \in P_1$.

LEMMA 1. *$(T_r, \{P_{1,(w_1,\cdots,w_k)}\})$ is an independent process.*

PROOF. This is intuitively clear because every Markov process can be considered as an independent process with time change. A rigorous proof is as follows: Let $M = (m_{ij})_{j=1,\cdots,n}^{i=1,\cdots,n}$ be the transition matrix for (T, P, X). By rescaling the partition so that $mP_1 = 1$, we get

$$m(P_{1,(w_1,\cdots,w_k)}) = m_{1w_1} m_{w_1w_2} \cdots m_{w_{k-1}w_k} m_{w_k1}.$$

For example, let

$$\begin{aligned} A &= \{x \mid x \in P_{1,(w_1,\cdots,w_k)}, T_r(x) \in P_{1,(v_1,\cdots,v_{k'})}, T_r^2(x) \in P_{1,(u_1,\cdots,u_{k''})}\} \\ &= \{x \in P_1 \mid T(x) \in P_{w_1}, T^2(x) \in P_{w_2} \cdots, T^k(x) \in P_{w_k}, T^{k+1}(x) \in P_1, \\ &\quad T^{k+2}(x) \in P_{v_1}, \cdots, T^{k+k'+1}(x) \in P_{v_{k'}}, T^{k+k'+2}(x) \in P_1, T^{k+k'+3}(x) \in P_{u_1}, \\ &\quad \cdots, T^{k+k'+k''+2}(x) \in P_{u_{k''}} T^{k+k'+k''+3}(x) \in P_1\}, \\ mA &= m_{1w_1} m_{w_1w_2} \cdots m_{w_k1} m_{1v_1} \cdots m_{v_{k'-1}v_{k'}} m_{v_{k'}1} m_{1u_1} \cdots m_{u_{k''-1}u_{k''}} m_{u_{k''}1} \\ &= m(P_{1,(w_1,\cdots,w_k)}) m(P_{1,(v_1,\cdots,v_k)}) m(P_{1,(u_1,\cdots,u_{k''})}). \end{aligned}$$

Since this holds for every name, it is clear that $(T_r, \{P_{1,(w_1,\cdots,w_l)}\})$ is an independent process.

(Note: This is true no matter what we take for a base set.)

LEMMA 2. *The flow under f' with $(T_r, \{P_{1,(w_1,\cdots,w_k)}\}, P_1)$ on a base is isomorphic to the given flow $(S_t, \bar{P}, X)$.*

PROOF. It's enough to show that every point except the set of measure zero in X is hit by a point x in P_1 once and only once until it comes back to P_1 under T. The ergodic theorem says that every point outside P_1 (except the set of measure zero) is hit at least once. If a point is hit more than once, then it has two different past names, which again belong to the set of measure zero. Hence there exists an obvious isomorphism between two flows.

These two lemmas enable us to prove the following theorem.

THEOREM 1. *A flow $(S_t, \bar{P}, \Omega)$ built under a step function with a multi-step Markov process (T, P, X) whose entropy is positive on the base is a direct product of a Bernoulli flow with a finite rotation flow.*

PROOF. If values of f' are irrationally related, then the flow $(S_t, \bar{P}, X)$ is Bernoulli (see [9]). If values of f' are rationally related, we can build this flow under a constant function g, with a multi-step Markov process on a base. Let (T', P', X') be this new multi-step Markov base. (T', P', X') is a direct product of a Bernoulli process and a rotation. Hence there exists a set $B \in \bigvee_0^k T^n P'$ for some k, which generates the rotation factor of (T', P', X'). Build the flow with the base B under the constant function g', whose value is an integer multiple of the value of g. Since (T', P', X') is a multi-step Markov process, there exists a partition Q of B such that Q is an independent partition under the induced map T_B on B and it generates the whole σ-algebra on X' with the partition $\{B, B^c\}$. In this picture, the rotation factor which sits independent of the Bernoulli factor is the obvious one. And the Bernoulli factor is the imbedded Bernoulli flow satisfying $S_{g'(x)} = T_B$ [5]. Clearly these two factors generate the whole σ-algebra of the flow. We note also that the rotation factor is spanned by finite continuous names.

COROLLARY 1. *If (T, P, X) is a 1-step Markov process and values of f are independent over the rationals, then $(S_t, \bar{P}, \Omega)$ is Bernoulli.*

PROOF. It is enough to prove that values of f' are irrationally related. Let $\{\alpha_1, \cdots, \alpha_n\}$ be values of f and $\{P_{\alpha_1}, \cdots, P_{\alpha_n}\}$ be the corresponding partition on a base. Since $E(T, P) > 0$, we may assume that $T(P_{\alpha_1}) = \{T(x) \mid x \in P_{\alpha_1}\}$ is not included in one of $\{P_{\alpha_i}\}_{i=1}^n$. If $T(P_{\alpha_1}) \cap P_{\alpha_1} \neq \emptyset$, then clearly values of f' are irrationally related.

Without loss of generality, let

$$T(P_{\alpha_1}) \cap P_{\alpha_2} \neq \emptyset \quad \text{and} \quad T(P_{\alpha_1}) \cap P_{\alpha_3} \neq \emptyset.$$

Let A, B be subsets of P_{α_1} satisfying $T(A) \subset P_{\alpha_2}$, $T(B) \subset P_{\alpha_3}$. For values of f' to be rationally related, they have to be the integer multiples of $\Sigma_{i=1}^n \alpha_i$. Hence every point $x \in A$ has to hit the set P_{α_3} before it comes back to P_{α_1} under T. Let k be the smallest integer such that $T^k(A) \cap P_{\alpha_3} \neq \emptyset$. Let A_0 be a subset of A satisfying $T^s(A_0) \subset P_{\alpha_{i_s}}$ for $s = 1, \cdots, k-1$ and $T^k(A_0) \subset P_{\alpha_3}$. Since $\Sigma_{s=1}^{k-1} \alpha_{i_s}$ is irrationally related to α_3, it follows that the values of f' on A_0 and those on B are irrationally related. Therefore values of f' are irrationally related.

COROLLARY 2. *If (T, P, X) is a mixing multi-step Markov process and values of f are independent over the rationals, then the flow $(S_t, \bar{P}, \Omega)$ is Bernoulli.*

PROOF. Divide P into a finer partition P' so that (T, P', X) is a 1-step mixing Markov process. Values of f, $\{\alpha_1, \cdots, \alpha_n\}$, over (T, P', X) are the same as the values of f except their multiplicity. Build the flow over P_1' as a base. It is enough to show that f' is irrationally related. P_1' has a partition $\{P'_{1,(w_1,\cdots,w_k)}\}$ according to path names. Let $k+1$ be the height of the path name $(w_1,\cdots,w_k)$. Since (T,P') is mixing, the heights do not have the g.c.d., i.e., there exist at least two heights a and b which are relatively prime.

Let $P'_{1,(w_1,\cdots,w_{a-1})}$ and $P'_{1,(v_1,\cdots,v_{b-1})}$ have heights a and b respectively.

$$f' = \begin{cases} \sum_{0}^{a-1} \alpha_{w_i} & \text{on } P'_{1,(w_1,\cdots,w_{a-1})} \\ \sum_{i=0}^{b-1} \alpha_{v_i} & \text{on } P'_{1,(v_1,\cdots,v_{b-1})} \end{cases} \qquad \text{where } w_0 = v_0 = 1.$$

If there exists w_i such that $\alpha_{w_i} \notin \{\alpha_{v_i}\}_{i=0}^{b-1}$ or α_{v_i} such that $\alpha_{v_i} \notin \{\alpha_{w_i}\}_{i=0}^{b-1}$, then clearly these two values are irrationally related. Otherwise let $\{\beta_1,\cdots,\beta_m\} = \{\alpha_{w_i}\}_{i=0}^{a-1} = \{\alpha_{v_i}\}_{i=0}^{b-1}$ where β_i's are independent over the rationals. Let

$$\sum_{i=0}^{a-1} \alpha_{w_i} = \sum_{j=1}^{m} r_j\beta_j \qquad \text{where } \sum_{j=1}^{m} r_j = a,$$

$$\sum_{i=0}^{b-1} \alpha_{v_i} = \sum_{j=1}^{m} s_j\beta_j \qquad \text{where } \sum_{j=1}^{m} s_j = b.$$

If $\Sigma_0^{a-1}\alpha_{w_i}$ and $\Sigma_{i=0}^{b-1}\alpha_{v_i}$ are rationally related, then there exist two integers l and l' such that where $(l,l')=1$,

$$l\left(\sum_{i=0}^{a-1} \alpha_{w_i}\right) = l'\left(\sum_{i=0}^{b-1} \alpha_{v_i}\right),$$

$$l\left(\sum_{j=1}^{m} r_j\beta_j\right) - l'\left(\sum_{j=1}^{m} s_j\beta_j\right) = 0,$$

$$\sum_{j=1}^{m} (lr_j - l's_j)\beta_j = 0,$$

$$lr_j - l's_j = 0 \quad \forall j \Rightarrow r_j = \frac{l'}{l} s_j \quad \forall j.$$

Since the r_j's are all integers and $(l,l')=1$, s_j/l's have to be integers for all j. Hence a and b have at least one common divisor, $\Sigma_{j=1}^{m} s_j/l$. This contradicts $(a,b)=1$.

COROLLARY 3. *A flow satisfying the Gurevič condition with a mixing multi-step Markov partition on the base is Bernoulli.*

PROOF. Gurevič proved that it is a K-flow [3]. Once it is a K-flow, it is Bernoulli by Theorem 1.

EXAMPLE. We cite an example where the function f takes values 1,1/2 and α (irrational number) with a mixing Markov partition on a base, but the flow is not Bernoulli. Let $P = \{P_0, P_1, \cdots, P_4\}$ be a partition on a base and M be a matrix representing T on a base.

$$f(x) = \begin{cases} 1 & \text{if } x \in P_0 \\ \alpha & \text{if } x \in P_1 \\ \alpha & \text{if } x \in P_2 \\ 1/2 & \text{if } x \in P_3 \\ 1/2 & \text{if } x \in P_3 \end{cases} \qquad M = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then clearly T is a mixing Markov transformation. If you build the flow over the base P_1, then f' takes two values $1+\alpha$ and $2+2\alpha$ which are certainly rationally related. Hence this flow has a rotation factor.

2. $\bar{d}$-limits

Let $(S_t^{(i)} \bar{P}^{(i)})$ be a $\bar{d}$-convergent sequence of flows with limit $(S_t, \bar{P})$. If there exists a subsequence $\{i_k\} \subset \{i\}$ such that $(S_t^{(i_k)}, \bar{P}^{(i_k)})$ is Bernoulli for all i_k, then $(S_t, \bar{P})$ is a Bernoulli flow. So we will now assume that none of these flows are Bernoulli, i.e., every flow is a direct product of a Bernoulli flow with a finite rotation. Denote the rotation number of $(S_t^{(i)}, \bar{P}^{(i)})$ by $r^{(i)}$ and the rotation factor by $R_r(i)$.

Let $(S_t, \bar{P})$ and $(S'_t, \bar{P}')$ be two flows with rotation factors R_r and $R_{r'}$ respectively, and $E[S_1] \geqq E[S'_1]$. Let $\tilde{\Omega}$ be a direct product of a Bernoulli flow of the entropy $E[S_1]$ with an ergodic joining of R_r and $R_{r'}$. Thus $(S_t, \bar{P})$ and $(S'_t, \bar{P}')$ occur as factors of $(\tilde{S}_t, \tilde{\Omega})$. If r and r' are irrationally related, then an ergodic joining of R_r and $R_{r'}$ is a direct product of R_r and $R_{r'}$. If they are rationally related, say $r = (p/q)r'$ where $(p,q) = 1$, then ergodic joining of these two is again a rotation whose rotation number is $q \cdot r = p \cdot r'$.

PROPOSITION 1. *Let $(S_t, \bar{P})$, $(S'_t, \bar{P}')$ and $(\tilde{S}_t, \tilde{\Omega})$ be as above with $E[S_1] < \infty$. Let $(\tilde{S}_t, \overset{()}{\bar{P}})$ be any embedding of $(S_t, \bar{P})$ in $(\tilde{S}_t, \tilde{\Omega})$. Then there exists an embedding $(\tilde{S}_t, \overset{()}{\bar{P}'})$* of *$(S'_t, \bar{P}')$ in $(\tilde{S}_t, \tilde{\Omega})$ such that*

$$|\overset{()}{\bar{P}'} - \overset{()}{\bar{P}}| < 16\sqrt{\bar{d}((S_t, \bar{P}), (S'_t, \bar{P}'))}.$$

PROOF. Ergodic joining of $(S_t, \bar{P})$ and $(S'_t, \bar{P}')$ that attains $\bar{d}((S_t, \bar{P}), (S_t, \bar{P}'))$ has an ergodic joining of R_r and $R_{r'}$ as a zero entropy factor. This zero entropy factor is independent of Bernoulli factors of $(S_t, \bar{P})$ and $(S'_t, \bar{P}')$ in the joining. Hence the proposition follows from the following lemma of A. Fieldsteel [2].

LEMMA 3. *Let S'_t be a flow on (Ω', F', μ') with S'_1 ergodic. Let $\bar{P}'$ and $\bar{Q}'$ be partitions of Ω' such that they generate F' under S'_1 and $(S'_1, (\bar{Q}')_{S_t})$ has a Bernoulli complement in (S'_1, F). Let S_t be a flow on (Ω, F, μ) with S_1 ergodic and $E[S_1] \geqq E[S'_1]$. Let $\bar{P}$ and $\bar{Q}$ be partitions of Ω such that $(S_t, \bar{Q}) \sim (S'_t, \bar{Q}')$ and $\bar{d}_{Q',Q}[(S_t, \bar{P}' \vee \bar{Q}'), (S_t, \bar{P} \vee \bar{Q})] < (\varepsilon/8)^2$. Then there exists a partition $\tilde{P}$ in Ω such that $|\bar{P} - \tilde{P}| < 2\varepsilon$ and $\bar{d}_{Q',Q}[(S'_t, \bar{P}' \vee \bar{Q}'), (S_t, \tilde{P} \vee \bar{Q})] = 0$.*

Since $\{(S_t^{(i)}, \bar{P}^{(i)})\}$ is a $\bar{d}$-convergent sequence of flows, the entropies converge. Therefore we can form a sequence of monotone entropy by replacing each space by its direct product with a Bernoulli flow of appropriate entropy, e_i. Now we apply the proposition repeatedly, with a subsequence $\{(S_t^{(i_k)}, \bar{P}^{(i_k)})\}$ satisfying $\Sigma e_{i_k} < \infty$, and obtain a $\bar{d}$-convergent embedding in (S'_t, Ω') which is a direct product of a Bernoulli flow of entropy $h = \sup\{E[S_1^{(i)}]\}$ with an ergodic joining of countably many rotation factors. Then the limit as a factor of (S'_t, Ω') is a direct product of a Bernoulli flow with a factor of an ergodic joining of all rotation factors.

PROPOSITION 2. *$\{(S_t^{(i)}, \bar{P}^{(i)})\}$ converges in $\bar{d}$ to a flow $(S_t, \bar{P})$ which has a finite rotation factor R_r. Then there exists a subsequence $\{i_k\}$ such that $\{r^{(i_k)}\}$ are rationally related to r.*

PROOF. If $(S_t, \bar{P})$ has a rotation factor R_r, then S_r is not ergodic. Since the $\bar{d}$-limit of ergodic processes is ergodic, there exists a subsequence $\{(S_t^{(i_k)}, \bar{P}^{(i_k)})\}$ such that none of $\{S_r^{(i_k)}\}$ is ergodic. $S_r^{(i_k)}$ is not an ergodic process only when the rotation number $r^{(i_k)}$ is rationally related to r.

Every $(S_t^{(i_k)}, \bar{P}^{(i_k)})$ is a direct product of a Bernoulli flow with a unique rotation factor $R_{r^{(i_k)}}$ where $r^{(i_k)}$ is rationally related to r. The $(S_t, \bar{P})$, as a $\bar{d}$-limit of these flows, is a factor of a direct product of a Bernoulli flow with an ergodic joining of rotation factors whose rotation numbers are rationally related.

COROLLARY 4. *Let $(S_t, \bar{P})$ be a $\bar{d}$-limit of a sequence $\{(S_t^{(i)}, \bar{P}^{(i)})\}$. If $(S_t, \bar{P})$ has a rotation factor R_r, then it cannot have other rotation factors whose rotation numbers are irrationally related to r.*

PROPOSITION 3. *Let $(S_t, \bar{P})$ be a flow under a step function. If it is a direct product of a Bernoulli flow with a flow of finite rotation, then canonical n-step*

Markov approximations converge in $\bar{d}$ if and only if the rotation factor is spanned by finite continuous names.

PROOF. ($\Rightarrow$) Let R_r be the rotation factor of $(S_t, \bar{P})$. If this rotation factor is not spanned by finite names, then canonical approximations don't have rotation factors, i.e., they are Bernoulli flows. If they converge in $\bar{d}$, then $(S_t, \bar{P})$ is a Bernoulli flow. Contradiction.

($\Leftarrow$) We need the following two lemmas. The first one is from Thouvenot's relatively finitely determined property [10].

LEMMA 4. *If $(T, P \vee H)$ is the direct product of a Bernoulli process and (T, H) with finite state space, then given $\varepsilon > 0$, there are integers m, n and a number δ so that if $(\bar{T}, Q \vee \bar{H})$ is any other ergodic process that satisfies $(T, H) = (\bar{T}, \bar{H})$, for all but δ of the atoms $E \in \bigvee_{-m}^{m} T^i H$*

$$\left| \operatorname{dist}\left(\bigvee_{-n}^{n} T^i P / E \right) - \operatorname{dist}\left(\bigvee_{-n}^{n} \bar{T}^i Q / E \right) \right| < \delta$$

and

$$| h(T, P \vee H) - h(\bar{T}, Q \vee \bar{H}) | < \delta,$$

then

$$\bar{d}((T, P), (\bar{T}, Q)) < \varepsilon.$$

The second lemma is from A. Fieldsteel [2].

LEMMA 5. *If $(S'_t, P' \vee Q')$ and $(S_t, P \vee Q)$ are continuous processes and N is an integer such that*

(1) $(Q')_{S'_t} = (Q')_{S'_1}$, $(Q)_{S_t} = (Q)_{S_1}$ *and* $(S'_t, Q') \sim (S_t, Q)$,

(2) $\| P' - \psi_N P' \|_{X, 1/N} < \eta / 3$ *and*

(3) $\bar{d}_{Q', Q} [(S'_{1/N}, P' \vee Q'), (S_{1/N}, P \vee Q)] < \eta / 3$,

then

$$\bar{d}_{Q', Q} [(S'_t, P' \vee Q'), (S_t, P \vee Q)] < \eta.$$

Since the rotation factor is spanned by finite names, there exists i_0 such that canonical Markov approximations $\{(S_t^{(i)}, \bar{P}^{(i)})\}$ have the same rotation factor for all $i \geqq i_0$.

PROPOSITION 4. *Let $(S_t, \bar{P})$ be a direct product of a Bernoulli flow with a finite rotation factor R_r. Then there is a flow $(S_t, \bar{P}')$ such that for given $\varepsilon > 0$*

(i) $\bar{d}((S_t, \bar{P}), (S_t, \bar{P}')) < \varepsilon$ *and*

(ii) *rotation factor is spanned by finite continuous names.*

(iii) *It is built under a function of finitely many values.*

PROOF. Since $(S_t, \bar{P})$ has a rotation factor R_r, we can consider the space to be a product space of a base (T, X) and an interval $[0, r)$. And the σ-algebra is the completed product σ-algebra. The sets in the partition $\bar{P} = \{\bar{P}_1, \cdots, \bar{P}_k\}$ are measurable with respect to this product σ-algebra. Hence we can approximate $\bar{P}$ as close as we want by measurable rectangles whose time axes are intervals. Also we can make the lengths of intervals rationally related to r; i.e., for given $\varepsilon > 0$ there exists $\bar{P}' = \{\bar{P}'_1, \cdots, \bar{P}'_k\}$ such that

(i) $\bar{P}'_i = \bigcup_{j=0}^{l} \bar{P}'_{ij}$ where $\{\bar{P}'_{ij}\}$ are measurable rectangles described as above,

(ii) $|\bar{P} - \bar{P}'| < \varepsilon$.

Since all the lengths of intervals are rationally related, $(S_t, \bar{P}')$ can be represented under a constant function f' with a base (T', X') where the value of f' is rationally related to r. If the rotation factor isn't spanned by finite names, then we can add a name to $(S_t, \bar{P}')$, within ε of $(S_t, \bar{P}')$ in $\bar{d}$, so that $(S_t, \bar{P}')$ together with the added name generates the rotation factor within finite time. So we might as well assume $(S_t, \bar{P}')$ satisfies all the conditions.

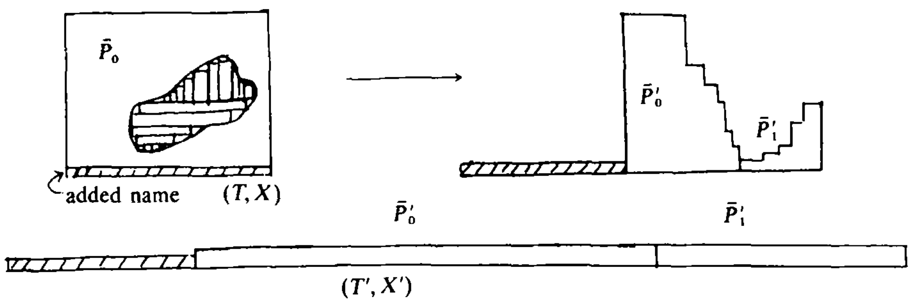

Fig. 1.

THEOREM 2. *The closure in $\bar{d}$ of all flows built under step functions with Markov partitions on bases is the set of flows, each being a direct product of a Bernoulli flow with a flow of rational pure point spectrum with respect to some number.*

PROOF. Flows of the form finite entropy Bernoulli X finite rotation are dense in Bernoulli X rational pure point spectrum with respect to some number. Therefore, the result follows from Propositions 3 and 4.

REFERENCES

1. R. Adler, P. Shields and M. Smorodinsky, *Irreducible Markov shifts*, Ann. Math. Stat. **43** (1972), 1027–1029.

2. A. Fieldsteel, *The relative isomorphism theorem for Bernoulli flows*, Dissertation of Berkeley, 1979.

3. B. M. Gurevič, *A special class of automorphisms and special flows*, Dokl. Akad. Nauk **4** (1964), 1738–1741.

4. D. S. Ornstein, *Bernoulli shifts with the same entropy are isomorphic*, Adv. Math. **4** (1970), 337–352.

5. D. S. Ornstein, *The isomorphism theorem for Bernoulli flows*, Adv. Math. **10** (1973), 124–142.

6. D. S. Ornstein, *Ergodic Theory, Randomness, and Dynamical Systems*, Yale University Press, 1974.

7. K. Park, Nice dense subsets of ergodic flows and Bernoulli flows, to appear.

8. D. J. Rudolph and G. Schwarz, *The limits in $\bar{d}$ of multi-step Markov chains*, Isr. J. Math. **28** (1977), 103–109.

9. P. Shields, *The Theory of Bernoulli Shifts*, Chicago University Press, Chicago, 1973.

10. J.-P. Thouvenot, *Quelques propriétés des systèmes dynamiques qui se décomposent en un produit de deux systèmes dont l'un est un schéma de Bernoulli*, Isr. J. Math. **21** (1975), 177–207.

Department of Mathematics
The Ohio State University
Columbus, OH 43210 USA